HW 6B.

$f \in \mathcal{BMF}(E)$: f is a bounded measurable function on E with $m(E) < +\infty$, say $f: E \to (m, M]$ with $m < M$ real. For any $n \in \mathbb{N}$ one divides $(m, M]$ into n-equal-length subintervals with partition points

$$y_0 = m < y_1 < y_2 < \cdots < y_{n-1} < y_n = M$$

Let $\varphi_n \in \mathcal{S}(E)$ and $\psi_n \in \mathcal{S}(E)$ be defined by

$$\varphi_n := \sum_{i=1}^{n} y_{i-1} \chi_{f^{-1}(y_{i-1}, y_i]},$$

$$\psi_n := \sum_{i=1}^{n} y_i \chi_{f^{-1}(y_{i-1}, y_i]}$$

Thus $\varphi_n \leq f \leq \psi_n$ & $\psi_n - \varphi_n \leq \frac{M-m}{n}$ $\forall n$

Define $\underline{\int}_E f := \sup\left\{\int_E \varphi : \varphi \in \mathcal{S}(E), \varphi \leq f \text{ a.e. on } E\right\}$

$\overline{\int}_E f := \inf\left\{\int_E \psi : \psi \in \mathcal{S}(E), f \leq \psi \text{ a.e. on } E\right\}$

(short as $\underline{\int} f$ & $\overline{\int} f$ if E is understood).

1. Show that $-\underline{\int} f = \overline{\int}(-f)$, $\underline{\int} f \leq \overline{\int} f$

and $\underline{\int} f$, $\overline{\int} f$ unchange if f is replaced

by some equivalent function. Show, moreover $\Delta_1, \Delta_2 \subseteq E$

2. $\underline{\int}(f+g) \geq \underline{\int} f + \underline{\int} g$, $\Delta_1 \cup \Delta_2 \in \mathfrak{M}$ then $\forall f, g \in \mathcal{BMF}(E)$

$\overline{\int}(f+g) \leq \overline{\int} f + \overline{\int} g$, $\int_{\Delta} f = \int_{\Delta_1} f + \int_{\Delta_2} f$

and $\underline{\int} f = \overline{\int} f$ (to be denoted by $\int_E f$ or $\int f$)

Hint: $f \chi_{\Delta} =$

$$\int \varphi_n \leq \underline{\int} f \leq \overline{\int} f \leq \int \psi_n$$

and $0 \leq \int \psi_n - \int \varphi_n = \int_E (\psi_n - \varphi_n) \leq \frac{M-m}{n} \cdot m(E)\ \forall n.$

3.* Show that $f \mapsto \int f$ is $\uparrow$ and linear

on $\mathcal{BMF}(E)$; and if $f \geq 0$ a.e. on E

and $\int_E f = 0$ then $f = 0$ a.e. on E

Let Δ be a measurable subset of E

Show that

$$\int_\Delta f = \int_E (f \chi_\Delta)$$

(Hint: via $\mathcal{S}(E)$ & $\mathcal{S}(\Delta)$).

Show moreover that if $\Delta := \Delta_1 \cup_0 \Delta_2 \subseteq E$ and $\Delta, \Delta_1, \Delta_2 \in \mathfrak{M}$ then $\int_\Delta f = \int_{\Delta_1} f + \int_{\Delta_2} f$.

(Hint: $f\chi_\Delta = f\chi_{\Delta_1} + f\chi_{\Delta_2}$)

4. Let $m(E) \le +\infty$, and

$f \in \mathcal{BMF}_0(E)$: bounded measurable function and $\exists\, A \subseteq E$ of finite measure s.t. $f = 0$ on $E \setminus A$. Define

$$\int_E f := \int_A f$$

Show that this definition is well-defined: if $B \subseteq E$, $m(B) < +\infty$ and $f = 0$ on $E \setminus B$ then $\int_A f = \int_{A \cup B} f = \int_B f$